J. Fluid Mech. (1972), *vol.* 55, *part* 1, *pp.* 113–127
Printed in Great Britain

Kelvin waves on oceanic boundaries

By JOHN W. MILES
Institute of Geophysics and Planetary Physics, University of California, La Jolla

(Received 5 May 1972)

The primitive Kelvin wave (on a rotating, semi-infinite, plane sheet of water of uniform depth bounded by a vertical wall) is corrected for the effects of the Earth's curvature, the reduction in depth over the continental shelf, and bends in the coastline. The results are of interest for coastal propagation of the tides; numerical examples are given for the California coastline. It is found that the Earth's curvature reduces the wave speed south of Cape Mendocino by 8–10 % (the possible range for other coastlines is roughly ± 15 %) and that the continental shelf reduces the wave speed by 2–8 %. The off-shore mass transport (which vanishes identically for the primitive Kelvin wave) induced by curvature and/or the shelf also is calculated. The analysis of Packham & Williams (1968) for diffraction of a Kelvin wave by a corner is extended to obtain explicit results for the phase of the transmission coefficient. It is found that a sustained change in the direction of the coastline may induce a phase shift of the order of an hour (1·3 hours for the bend at Cape Mendocino), but that small distortions of the coastline without a sustained change in direction have negligible effects on the transmitted Kelvin wave at tidal frequencies.

1. Introduction

The classical Kelvin wave (Lamb 1932, § 208) is a trapped solution of the equations governing small disturbances on a rotating, semi-infinite, plane sheet of water of uniform depth h_1 that is bounded by a vertical wall. It advances along this boundary with the phase speed

$$c_1 = (gh_1)^{\frac{1}{2}} \tag{1.1}$$

in the direction of rotation (south on a western boundary or north on an eastern boundary in the northern hemisphere). We consider here the necessary modifications of this model to allow for the Earth's curvature, the reduction in depth over a continental shelf, and the departure from a straight coastline, all on the hypothesis that these individual effects are either sufficiently small or sufficiently independent to permit their linear superposition. The results are of interest primarily for coastal propagation of the tides (cf. Munk, Snodgrass & Wimbush 1970), and we therefore restrict our considerations to frequencies of the order of the Earth's rotational speed.

The fact that disturbances which closely resemble Kelvin waves are observed in terrestial oceans suggests that the Earth's curvature must have only secondary effects on the classical Kelvin wave at tidal frequencies. We give a first-order

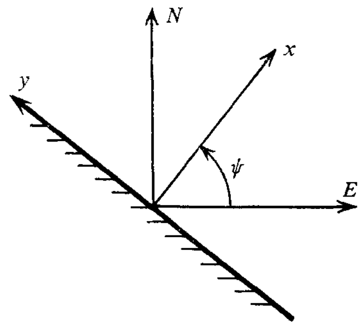

FIGURE 1. The Mercator and coastal co-ordinates; see (2.10). The line $x = 0$ is the straight coastline of § 3.

asymptotic calculation of these effects in § 3 for middle latitudes and sufficiently large values of the dimensionless wavenumber

$$\ell_1 = \sigma a / c_1, \tag{1.2}$$

where σ is the angular frequency and a is the radius of the Earth. The end result, (3.11) below, implies that the amplitude of the modified Kelvin wave is proportional to the square root of the sine of the latitude (a result obtained earlier by Moore (1968) for an equatorial Kelvin wave) and that its wave speed is

$$c \doteqdot c_1 + \tfrac{1}{4} c_1^2 (\Omega a)^{-1} \sec\theta \sin\psi \quad (\Omega a \cos\theta \gg c_1), \tag{1.3}$$

where Ω is the rotational speed of the Earth, θ is the latitude, and ψ is the angle shown in figure 1. The change in wave speed is -8–10 % along the California coast south of Cape Mendocino ($\psi = 230°$) and as large as -15 % on the south coast of Australia.

Munk *et al.* (1970) calculate the reduction of the wave speed for both step and exponential shelves. Smith (1972) calculates the corresponding reduction for an arbitrary profile through a perturbation expansion (see also § 4 below), taking advantage of the fact that the shelf width is small relative to the tidal wavelength. The end result is

$$c \doteqdot c_1 - 2\Omega d \sin\theta \quad (\Omega d \ll c_1), \tag{1.4a}$$

where

$$d = \int_0^\infty \{1 - (h/h_1)\}\, d\hat{x}, \quad h \sim h_1 \quad (\hat{x} \to \infty), \tag{1.4b,c}$$

h is the local depth, and $\hat{x}$ is the off-shore co-ordinate. The reduction for the California coast ranges from a maximum of 8 % for the broad shelf off Southern California ($d \doteqdot 190$ km, $h_1 = 4$ km) to a minimum of roughly 2 % for the relatively narrower shelf north of Cape Mendocino ($d \doteqdot 50$ km, $h_1 = 3$ km).

The effect of a sharp corner on a Kelvin wave has been calculated by Packham & Williams (1968); however, they give explicit results only for the amplitude of the diffracted Kelvin wave. We deduce the corresponding phase shift from their results in the appendix and give an independent calculation for small angles in § 4 (which is perhaps warranted by the complicated nature of the analysis of Packham & Williams). The asymptotic time delay owing to a concave corner of angle $\pi - \epsilon$ (figure 2(*b*)) is

$$t_d = -\tfrac{1}{4}\epsilon(\Omega \sin\theta)^{-1} \quad (|\epsilon| \ll 1), \tag{1.5}$$

which is between one and two hours for a 45° corner in middle latitudes. We also show, in § 4, that the effect of a bump of amplitude b, without a sustained change in direction, is of the order of $(\sigma b/c)^2$, so that small distortions in the coastline are negligible in the present context.

2. Equations of motion

We begin with the Laplace tidal equations in homogeneous form (Lamb 1932, § 214)

$$\hat{\mathbf{v}}_t + 2\boldsymbol{\Omega} \times \hat{\mathbf{v}} + g\nabla\hat{\zeta} = 0 \tag{2.1a}$$

and

$$\nabla . (h\hat{\mathbf{v}}) + \hat{\zeta}_t = 0, \tag{2.1b}$$

where $\hat{\mathbf{v}}$ is the horizontal velocity, $\hat{\zeta}$ is the vertical displacement of the free surface, $\boldsymbol{\Omega}$ is the angular velocity of the Earth (only the vertical component of which is significant here), g is the acceleration of gravity, h is the local depth, t is time, and subscripts imply partial differentiation. Referring $\hat{\zeta}$ to the depth h_1 and $\hat{\mathbf{v}}$ to the corresponding gravity-wave speed, equation (1.1), and assuming an harmonic time dependence with angular frequency σ, we pose the solution to (2.1) in the form

$$\{\hat{\zeta}, \hat{\mathbf{v}}\} = \mathscr{R}[\{h_1\zeta(\lambda,\mu), c_1\mathbf{v}(\lambda,\mu)\}e^{i\sigma t}], \tag{2.2}$$

where $\mathscr{R}$ implies *the real part of*, ζ and v are dimensionless complex amplitudes, and λ and μ are Mercator co-ordinates; λ (positive east) is the conventional longitude, whilst μ (positive north) is related to the conventional latitude θ through the transformation

$$\cos\theta = \operatorname{sech}\mu, \quad \sin\theta = \tanh\mu. \tag{2.3}$$

The differential displacement on a spherical Earth of radius a and the gradient operator then are given by

$$d\mathbf{r} = (a \operatorname{sech}\mu)\,(d\lambda, d\mu) \tag{2.4a}$$

and

$$\nabla = (a \operatorname{sech}\mu)^{-1}(\partial_\lambda, \partial_\mu), \tag{2.4b}$$

where, here and subsequently, the couplet (A, B) is a vector having the λ-component A and the μ-component B. We also introduce

$$f = (2\Omega/\sigma)\tanh\mu \equiv f_1 \tanh\mu \tag{2.5}$$

as a dimensionless Coriolis parameter,

$$h = h/h_1 \tag{2.6}$$

as a dimensionless depth, and

$$k = k_1 \operatorname{sech}\mu \tag{2.7}$$

as a dimensionless wavenumber, where k_1 is given by (1.2).

Substituting (2.2)–(2.7) into (2.1), we obtain

$$\mathbf{v} = (i/k)(1-f^2)^{-1}(\zeta_\lambda - if\zeta_\mu, \zeta_\mu + if\zeta_\lambda) \tag{2.8}$$

and

$$\left[\left(\frac{h}{1-f^2}\right)(\zeta_\lambda - if\zeta_\mu)\right]_\lambda + \left[\left(\frac{h}{1-f^2}\right)(\zeta_\mu + if\zeta_\lambda)\right]_\mu + k^2\zeta = 0. \tag{2.9}$$

We seek the solution of (2.9) in the neighbourhood of an approximately straight coastline, say $x = x_0(y)$, $|x_0'| \ll 1$, where x and y are defined by the rotation (see figure 1)

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{Bmatrix} \lambda - \lambda_0 \\ \mu - \mu_0 \end{Bmatrix} \quad (c \equiv \cos\psi, \quad s \equiv \sin\psi). \tag{2.10}$$

The differential operator in (2.9) is invariant under this rotation, whence

$$\left[\left(\frac{h}{1-f^2}\right)(\zeta_x - if\zeta_y)\right]_x + \left[\left(\frac{h}{1-f^2}\right)(\zeta_y + if\zeta_x)\right]_y + k^2\zeta = 0. \tag{2.11}$$

The boundary condition that there be no mass transport across the coastline transforms to

$$h(\zeta_x - if\zeta_y) = 0 \quad (x = x_0); \tag{2.12a}$$

in addition, ζ must satisfy the finiteness condition

$$\zeta \to 0 \quad (x \to \infty). \tag{2.12b}$$

We emphasize that the length scale for x and y is $a \operatorname{sech} \mu$.

3. Effects of Earth's curvature

We isolate the effects of the Earth's curvature on a Kelvin wave by setting $h \equiv 1$ (uniform depth, $h \equiv h_1$) and $x_0 \equiv 0$ in (2.11) and (2.12) to obtain

$$\frac{\zeta_{xx} + \zeta_{yy}}{1-f^2} + \left(\frac{s + icf}{1-f^2}\right)' \zeta_x + \left(\frac{c - isf}{1-f^2}\right)' \zeta_y + k^2\zeta = 0 \tag{3.1}$$

and

$$\zeta_x - if\zeta_y = 0 \quad (x = 0), \qquad \zeta \to 0 \quad (x \to \infty), \tag{3.2a,b}$$

where, throughout this section, the primes imply differentiation with respect to μ.

The solution of (3.1) and (3.2) for constant f and k is given by the (primitive Kelvin wave)

$$\zeta(x, y) = A\, e^{-fkx + iky} \quad (f' = k' = 0), \tag{3.3}$$

where A is a constant amplitude. Guided by this result and by Green's asymptotic solution of the problem of one-dimensional wave propagation in a medium of slowly varying properties, we seek an asymptotic solution of (3.1) and (3.2) for $k \to \infty$ with $kx = O(1)$ in the form

$$\zeta(x, y) = A(x, y) \exp\left\{-f_0 k_0 x + i\int_0^y k_0\, dy\right\}, \tag{3.4}$$

where

$$f_0 \equiv f(\mu_0 + cy) \quad \text{and} \quad k_0 \equiv k(\mu_0 + cy) \tag{3.5}$$

are evaluated at $x = 0$, μ_0 is a reference latitude, and $A(x, y)$ is, by hypothesis, a slowly varying function *vis-à-vis* the exponential.

Being interested primarily in the neighbourhood of the coastline, we do not require uniform validity of (3.4) as $x \to \infty$ and may expand $A(x,y)$ about $x = 0$. Requiring (3.4) to satisfy (3.2*a*, *b*), we find that an appropriate form of this expansion is

$$A(x,y) = A_0(y) + i f_0 x (dA_0/dy) + \tfrac{1}{2}x^2 A_2(y) + O(1/k_0^2), \quad k_0 x = O(1) \quad (k_0 \to \infty), \tag{3.6}$$

in which (we anticipate) $A_2 = O(k_0)$. Multiplying (3.1) through by $1-f^2$, expanding f and f' about $x = 0$, substituting (3.4) and (3.6) into the result, and retaining only the dominant terms, which are $O(k_0)$ as $k_0 \to \infty$ with $k_0 x = O(1)$, we obtain

$$2ik_0(1-f_0^2)(dA_0/dy) + [ik_0' c + k_0(s + if_0 c) f_0' - 2ik_0(f_0 k_0)' cx + \{(1-f_0^2) k_0^2\}' sx] A_0 + (1 - 2f_0 k_0 x) A_2 = 0. \tag{3.7}$$

Equating the terms of degree zero and one in x separately to zero, we obtain

$$A_2 = f_0^{-1}\{k_0' s - (ic + f_0 s)(f_0 k_0)'\} A_0 \tag{3.8}$$

and

$$(dA_0/dy) - \tfrac{1}{2}\{(f_0'/f_0) c + i(k_0'/f_0 k_0) s\} A_0 = 0. \tag{3.9}$$

Invoking (2.5) and (2.7) to simplify $(k_0'/f_0 k_0)$ in (3.9), and then integrating, we obtain

$$A_0(y) = A_0(0)\{f(\mu_0 + cy)/f(\mu_0)\}^{\frac{1}{2}} \exp\{-\tfrac{1}{2}i(s/f_1)y\}. \tag{3.10}$$

Setting $x = 0$ in (3.4) and invoking (3.6) and (3.10), we obtain

$$\zeta(0,y) = \zeta(0,0)\{f(\mu_0 + cy)/f(\mu_0)\}^{\frac{1}{2}} \exp\left[i\int_0^y \{k - \tfrac{1}{2}(s/f_1)\}\, dy\right] \tag{3.11†}$$

within an error factor of $1 + O(1/k_0^2)$. We conclude that, to first order in $1/k_0$, the principal effects of the Earth's curvature on a Kelvin wave, moving along a straight coastline running ψ west of north (see figure 1), are to alter its amplitude in proportion to $f^{\frac{1}{2}}$ or, equivalently, $(\sin\theta)^{\frac{1}{2}}$, and its reciprocal wave speed from $1/c_1$ to

$$c^{-1} = (\sigma a \operatorname{sech}\mu)^{-1}\{k - \tfrac{1}{2}(s/f_1)\} \tag{3.12a}$$

$$= c_1^{-1} - \tfrac{1}{4}(\Omega a)^{-1} \sec\theta \sin\psi. \tag{3.12b}$$

The increase in amplitude between 30° N and 40° N on a western boundary is 13 %. The decrease in wave speed for $\theta = 35°$, $\psi = 230°$, and $h_1 = 4$ km (corresponding to the California coastline south of Cape Mendocino) is 10 %.

The variation of the Kelvin-wave amplitude with latitude is a direct consequence of conservation of energy (this is typical of Green's approximation; cf. Lamb (1932, § 186)) and reflects the fact that the thickness of the Kelvin-wave boundary layer (after allowing for the Mercator scaling) is inversely proportional to f. The variation of the wave speed is an indirect consequence of the variation of the metrical coefficient in a direction normal to the coastline and is curious in being inversely proportional to the variation of the Coriolis parameter with latitude. We remark that the invocation of Rossby's beta-plane approximation,

† If $k = k(y)$ is slowly varying, the right-hand side of (3.11) should be multiplied by $\{k(0)/k(y)\}^{\frac{1}{2}}$ and k replaced by $k\{k(0)/k(y)\}^{\frac{1}{2}}$ in the integrand.

in which f, f' and k are regarded as constant in (3.1) and terms of order f'^2 are neglected, implies

$$k+\tfrac{1}{2}s(1-f^2)^{-1}f'$$

in place of $k-\tfrac{1}{2}(s/f_1)$ in (3.12*a*), whence

$$c^{-1} = c_1^{-1}+(\Omega/a)\,(\sigma^2-4\Omega^2\sin^2\theta)^{-1}\cos\theta\sin\psi \quad (f, f', k \text{ constant}). \quad (3.13)$$

This approximation yields a correction that is exactly the negative of that provided by (3.12*b*) if $\sigma = 2\Omega$ (semi-diurnal tides), and it typically yields a much larger change in wave speed (with a singularity at $\theta = 30°$) if $\sigma = \Omega$ (diurnal tides). In fact, the beta-plane approximation is inconsistent in the present context, in which the variation of k^2 in (3.1) implies effects that are comparable with those implied by f'.

The particle velocity accompanying a primitive Kelvin wave is parallel to the coastline. A convenient measure of the departure of the velocity field from this primitive state is the *ellipticity* (cf. Munk *et al.* 1970)

$$e \equiv -i(\zeta_x - if\zeta_y)/(\zeta_y+if\zeta_x). \quad (3.14)$$

Substituting (3.4)–(3.6) into (3.14), invoking (3.8) and (3.9), and retaining only the dominant terms, we obtain

$$e = (f_0k_0)^{-1}\{i(f_0k_0)'\,c - sk_0'\}\,x \quad (k_0x \ll 1) \quad (3.15a)$$

$$= \{2i\cos\psi\cot 2\theta + f_1^{-1}\sin\psi\sec\theta\}\,(\hat{x}/a), \quad (3.15b)$$

where $\hat{x}$ is the dimensional off-shore distance. The number in curly brackets is $-(0{\cdot}94+i0{\cdot}44)$ for the semi-diurnal tide ($f_1 = 1$) on the California coast south of Cape Mendocino. The dimensionless, complex amplitude of the offshore mass transport is, from (2.8) and (3.14),

$$m = (i/k)\,(1-f^2)^{-1}\,(\zeta_x - i\zeta_y)\,h \doteq -ieh\zeta \quad (3.16)$$

in first approximation ($h \equiv 1$ here, but not in § 5 below).

4. Shelf effects

We calculate the effect of the variation in depth over the continental shelf on the assumption that the shelf width is small compared with each of the other characteristic lengths for the Kelvin wave. A convenient measure of the shelf width (suggested by the displacement thickness of boundary-layer theory) is

$$\delta = \int_0^\infty (1-h)\,dx, \quad (4.1)$$

in terms of which the above assumption implies

$$\delta, |\delta'(y)| \ll 1, 1/k; \quad |x_0'(y)|\,\delta \ll 1. \quad (4.2)$$

We then may neglect the local variation of each of δ, f and k with respect to y and set $h = h(x)$ and $x_0 = 0$ in calculating shelf-width effects (note, however, that k, as defined by (2.7), is referred to the outer depth h_1). Invoking these approximations in (2.11), we obtain

$$(h\zeta_x)_x + h\zeta_{yy} - i f h_x \zeta_y + (1-f^2) k^2 \zeta = 0. \tag{4.3}$$

Guided by the known solution for $h = 1$, equation (3.3), we seek a solution to (4.3) in the form

$$\zeta(x,y) = A(x)\, e^{-f\beta x + i\beta y}. \tag{4.4}$$

Substituting (4.4) into (4.3) and (2.12), we obtain

$$(e^{-2f\beta x} hA')' + (1-f^2)(k^2 - \beta^2 h)\, e^{-2f\beta x} A = 0 \tag{4.5}$$

and $$hA' = 0 \quad (x = 0), \qquad A = o(e^{f\beta x}) \quad (x \to \infty), \tag{4.6a, b}$$

where primes now imply differentiation with respect to x.

Integrating (4.5) over $(0, \infty)$ and invoking (4.6), we obtain

$$\int_0^\infty (k^2 - \beta^2 h)\, e^{-2f\beta x} A(x)\, dx = 0. \tag{4.7}$$

Substituting the primitive approximation

$$A(x) = A_0\{1 + O(k\delta)\}, \quad A_0 \equiv A(0), \tag{4.8a, b}$$

into (4.7) and invoking the fact that $h = 1$ for $x \gg \delta$, we obtain

$$\left\{(k^2 - \beta^2)(2f\beta)^{-1} + \beta^2 \int_0^\infty (1-h)\, dx\right\}\{1 + O(k\delta)\} = 0, \tag{4.9}$$

which (together with the restriction $\beta > 0$) implies

$$\beta = k\{1 + fk\delta + O(k^2\delta^2)\}. \tag{4.10}$$

The wave speed implied by (4.4) and (4.10) on $x = 0$ is†

$$c = c_1\{1 - fk\delta + O(k^2\delta^2)\} \tag{4.11a}$$

$$= c_1 - 2\Omega d \sin\theta + O(k^2\delta^2 c_1), \tag{4.11b}$$

where d, the dimensional counterpart of δ, is defined by (4.1) with x replaced by the dimensional distance from the coastline therein. Numerical evaluations of δ, based on measured profiles, yield $fk\delta = 0{\cdot}08$ for the broad shelf off Southern California and $0{\cdot}02$ for the narrowest portions of the shelf near Cape Mendocino. The former value compares with the estimate of $0{\cdot}06$ for either a two-step or an exponential fit to the Southern California shelf (Munk *et al.* 1970).

Substituting (4.8a) and (4.10) into (4.5) and invoking the restriction $x = O(\delta)$, we obtain

$$(hA')' = -(1-f^2)k^2(1-h)A_0\{1 + O(k\delta)\} \quad (x = O(\delta)). \tag{4.12}$$

† This result is anticipated by Smith (1972), whose paper appeared while the present paper was *sub judice*. His derivation differs somewhat from that given here, and he does not derive counterparts of (4.13) and (4.14); accordingly, I have thought it worthwhile to present the above analysis as originally written.

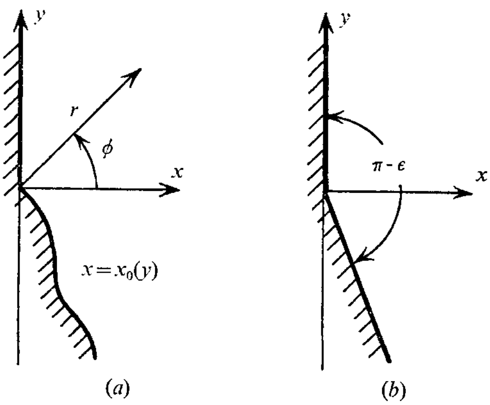

FIGURE 2. (*a*) An arbitrary distortion of an approximately straight coastline. (*b*) A corner.

Integrating (4.12) subject to (4.6*a*) and (4.8*b*), we obtain

$$A(x) = A_0\left\{1-(1-f^2)k^2\int_0^x \frac{dx}{h}\int_0^x (1-h)\,dx + O(k^3\delta^3)\right\}. \tag{4.13}$$

Substituting (4.4) and (4.13) into (3.14), we obtain

$$e = kh^{-1}\int_0^x (1-h)\,dx + O(k^2\delta^2), \tag{4.14}$$

which is $O(k\delta)$ for a given profile but may be numerically much larger than $k\delta$ in shallow water. (The fact that e is finite at $x = 0$ if h vanishes linearly as $x \downarrow 0$ reflects the singularity in the equations of motion at $h = 0$.)

The approximations in the preceding paragraph are not uniformly valid as $kx \to \infty$. Uniformly valid approximations may be obtained by replacing (4.4) by

$$\zeta(x,y) = A(x)\,e^{-\mu x + i\beta y}, \quad \mu = \{\beta^2-(1-f^2)\,k^2\}^{\frac{1}{2}}, \tag{4.15a,b}$$

and transforming the resulting modifications of (4.5) and (4.6) to an integral equation, which may be solved by iteration (cf. Lighthill 1957, §6) to obtain both β and $A(x)$. This procedure yields approximations that are equivalent to (4.10) and (4.13), but it is rather less efficient than the procedure followed here.

5. Bend in coastline

We now consider a coastline that departs slightly from $x = 0$ in $y < 0$ (see figure 2(*a*)), such that

$$x_0(y) \equiv 0 \quad (y \geqslant 0), \qquad |x_0'(y)| \ll 1 \quad (y < 0). \tag{5.1}$$

Proceeding as in thin-airfoil theory, we perturb the Kelvin wave

$$\zeta^{(0)}(x,y) = A\,e^{-fkx+iky} \tag{5.2}$$

by placing a distributed source, with strength proportional to $x_0'(y)$, along $y < 0$ and projecting the boundary condition (2.12*a*) on $x = 0$. Solving the resulting boundary-value problem with the aid of Fourier transformation over y or, equivalently, by generalizing Buchwald's (1971) result for a point source on a straight coastline, we obtain

$$\zeta(x,y) = \zeta^{(0)}(x,y) + \frac{(1-f^2)kA}{2\pi i}\int_{-\infty}^{0} x_0'(\eta)\,e^{ik\eta}\,d\eta \int_{-\infty}^{\infty} \frac{e^{-\mu x + i\beta(y-\eta)}\,d\beta}{\mu - f\beta}, \tag{5.3}$$

where
$$\mu = \{\beta^2 - (1-f^2)k^2\}^{\frac{1}{2}} \quad (\mathscr{R}\mu \geqslant 0). \tag{5.4}$$

The simplest and most interesting special case is a corner, $x_0' = -\epsilon$, where $\epsilon > 0$ if the corner is concave (see figure 2(*b*)). Carrying out the η integration in (5.3) for this case yields

$$\zeta(x,y) = \zeta^{(0)}(x,y) + \frac{\epsilon(1-f^2)kA}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-\mu x + i\beta y}\,d\beta}{(k-\beta)(\mu - f\beta)}. \tag{5.5}$$

Carrying out an asymptotic approximation by the method of steepest descent (the calculation differs from that given by Buchwald only in that the pole at $\beta = k$ is now double), we obtain

$$\zeta(x,y) \sim \zeta_K(x,y) + \zeta_P(x,y), \tag{5.6a}$$

where
$$\zeta_K(x,y) = \zeta^{(0)}(x,y)\,[1 + i\epsilon(-kx + ifky + \tfrac{1}{2}f^{-1})\,H(\phi_f - \phi)], \tag{5.6b}$$

$$\zeta_P(x,y) = \frac{\epsilon A(1-f^2)\,e^{-i(\kappa r - \frac{1}{4}\pi)}\cos\phi}{(2\pi\kappa r)^{\frac{1}{2}}\{1 + (1-f^2)^{\frac{1}{2}}\sin\phi\}(f\sin\phi + i\cos\phi)} + O\{(\kappa r)^{-\frac{3}{2}}\}, \tag{5.6c}$$

$$\kappa = (1-f^2)^{\frac{1}{2}}k, \quad \phi_f = -\sin^{-1}(1-f^2)^{\frac{1}{2}}, \tag{5.7a,b}$$

and r and ϕ are cylindrical polar co-ordinates (see figure 2). If $f > 1$, $\phi_f \equiv 0$, and ζ_P, the Poincaré wave, decays exponentially in $|\kappa|\,r$. Rewriting the term in square brackets as an exponential, which we may do without altering the magnitude of the error implicit in (5.3), we obtain

$$\zeta_K(x,y) = \zeta^{(0)}(x+\epsilon y, y - \epsilon x)\exp(\tfrac{1}{2}i\epsilon/f)\{1 + O(\epsilon^2)\} \quad (\phi < \phi_f), \tag{5.8}$$

which represents a Kelvin wave moving parallel to $x = -\epsilon y$ in $\phi < \phi_f$ with a phase advance of $\frac{1}{2}\epsilon/f$ relative to $\zeta^{(0)}$. The corresponding delay time is

$$t_d = -\tfrac{1}{2}\epsilon(f\sigma)^{-1} = -\tfrac{1}{4}\epsilon(\Omega\sin\theta)^{-1}. \tag{5.9}$$

The amplitude of ζ_K, which differs from that of $\zeta^{(0)}$ by $O(\epsilon^2)$, may be calculated through an energy balance. The dimensionless energy flux (referred to $\frac{1}{4}\rho g h^2$ as the unit of energy per unit area) in the incident Kelvin wave, for which the dimensionless group velocity is unity, is given by

$$\mathscr{E}^{(0)} = \int_0^\infty (|\zeta^{(0)}|^2 + |v^{(0)}|^2)\,dx = 2\int_0^\infty |\zeta^{(0)}|^2\,dx = (fk)^{-1}|A|^2. \tag{5.10}$$

A similar result holds for the transmitted Kelvin wave. The corresponding energy flux for the Poincaré wave, for which the dimensionless group velocity is $(1-f^2)^{\frac{1}{2}}$, is

$$\mathscr{E}_P = (1-f^2)^{\frac{1}{2}}\int_0^\pi \left\{|\zeta_P|^2 + \left(\frac{1+f^2}{1-f^2}\right)|\zeta_P|^2\right\} r\,d\phi \quad (\kappa r \to \infty) \tag{5.11a}$$

$$= \tfrac{1}{4}\epsilon^2(f^3k)^{-1}(1-f^4)\,|A|^2 H(1-f)\{1 + O(\epsilon^2)\}, \tag{5.11b}$$

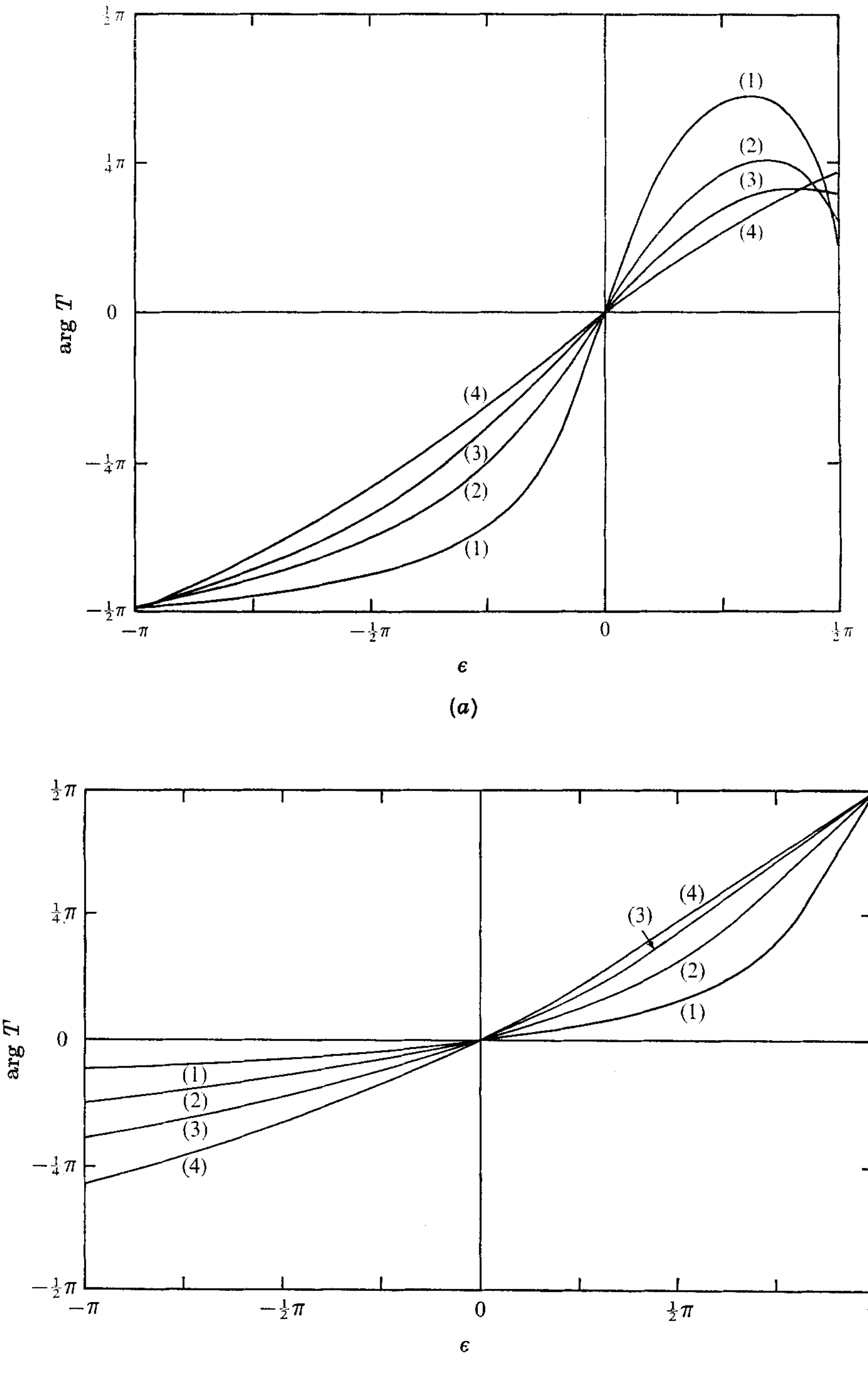

FIGURE 3. The phase of the Kelvin-wave transmission coefficient for the corner of figure 2 (*b*). (*a*) For $f = \frac{1}{5}n$, $n = 1, 2, 3, 4$. (*b*) For $1/f = \frac{1}{5}n$, $n = 1, 2, 3, 4$. These same curves also give q for $f = \frac{1}{5}n$.

where (5.6*c*) has been substituted into (5.11*a*) to obtain (5.11*b*); $\mathscr{E}_P \equiv 0$ for $\ell > 1$. The amplitude of the transmission coefficient then is given by

$$|T| \equiv |\zeta_K/\zeta^{(0)}| = \{1-(\mathscr{E}_P/\mathscr{E}^{(0)})\}^{\frac{1}{2}} = 1-\tfrac{1}{8}\epsilon^2(\ell^{-2}-\ell^2)\,H(1-\ell), \qquad (5.12)$$

in agreement with the exact result (Packham & Williams 1968; (A 6) below) to $O(\epsilon^2)$. The corresponding approximation implied by (5.8) is

$$\arg T \equiv \arg\{\zeta_K/\zeta^{(0)}\} = \tfrac{1}{2}\epsilon/\ell + O(\epsilon^2). \qquad (5.13)$$

The exact results (see appendix) for $\arg T$ are plotted in figure 3. The exact results for $|T|$ are plotted by Packham & Williams (1968). The approximate results, (5.12) and (5.13), are roughly adequate for $\ell > \frac{1}{2}$ and $|\epsilon| < \frac{1}{4}\pi$.

The results (5.6)–(5.13) hold for any bend (sustained change of direction) in the coastline that takes place within a coastal distance that is small compared with $1/k$ (this conclusion rests on the asymptotic development of the integral in (5.3) and is not necessarily valid for non-small ϵ). If the change of direction is not sustained but there is a sustained displacement, such that $x_0 \equiv x_1$ for $y < -l$, we may integrate (5.3) by parts to obtain

$$\zeta(x,y) = \zeta^{(0)}(x,y) + \frac{(1-\ell^2)\,kA}{2\pi}\left\{ix_1\, e^{-ikl}\int_{-\infty}^{\infty}\frac{e^{-\mu x+i\beta(y+l)}\,d\beta}{\mu-\ell\beta} - k\int_{-l}^{0}x_0(\eta)\,e^{ik\eta}\,d\eta\int_{-\infty}^{\infty}\frac{(k-\beta)\,e^{-\mu x+i\beta(y-\eta)}\,d\beta}{\mu-\ell\beta}\right\}. \qquad (5.14)$$

Carrying out the asymptotic approximation as above, we obtain

$$\zeta(x,y) \sim \zeta^{(0)}(x-x_1,y)\{1+O(k^2x_1^2)\} \quad (ky \to -\infty). \qquad (5.15)$$

It follows that the change of phase induced in the Kelvin wave by a displacement of the coastline (without a sustained change of direction) must be $O(k^2x_1^2)$. The same is true for a bump without a sustained displacement: if $x_0(y)$ is continuous, $x_0 = 0$ in $y \leqslant -l$ and $|x_0| \leqslant x_1$ in $0 \geqslant y \geqslant -l$, then only the second integral in the braces of (5.14) remains after integrating (5.3) by parts, and

$$\zeta(x,y) \sim \zeta^{(0)}(x,y)\{1+O(k^2|x_0|^2_{\max})\} \quad (ky \to -\infty). \qquad (5.16)$$

This last result is related to that for a harbour of (dimensionless) area A, for which the change in phase of the diffracted Kelvin wave is $O(\ell k^2 A)$ (Miles 1972).

6. California coastline

Combining the curvature and shelf effects of §§ 3 and 4, we find that the net speed of a Kelvin wave along the California coast varies from roughly $0{\cdot}82c_1$ south of, to $0{\cdot}98c_1$ north of, Cape Mendocino. Munk *et al.* (1970) report that the M_2 (semi-diurnal) tide advances northward with a speed of $0{\cdot}7c_1$ along the California coast south of Cape Mendocino and construct a semi-empirical model for this coastal tide in which the dominant component is a Kelvin wave that is corrected for the shelf but not for the Earth's curvature. It seems likely that the latter correction would increase the dominance of the Kelvin wave (*vis-à-vis* the free Poincaré wave and the forced waves) in their model.

The direction of the California coastline *at the outer edge of the continental shelf* is roughly constant ($\psi = 230°$) from Punta Baja (30° N) to Cape Mendocino (40° N), at which point it turns through 50° and runs approximately due north as far as Juan de Fuca Strait (48° N). The asymptotic time delay for a Kelvin wave turning the corner at Cape Mendocino, using either the approximation (5.9) or the exact result (see appendix), is 1·3 h.† This is in rough agreement with the observed (Munk *et al.* 1970) anomalies in the M_2 tide between Point Arguello and Crescent City; however, these anomalies are observed south, as well as north, of Cape Mendocino and therefore cannot all be charged to Kelvin-wave diffraction at the corner. [The change in depth across the Mendocino fracture zone also induces a time delay in the Kelvin wave, but an unpublished calculation reveals that this delay is small compared with that induced by the bend.]

This work was partially supported by the National Science Foundation under grant NSF-GA-10324, and by the Office of Naval Research, under Contract N00014-69-A-0200-6005.

Appendix. Transmission coefficient for corner

Packham & Williams (1968, § 5) obtain the complex Kelvin-wave transmission coefficient for a corner of internal angle $\gamma \equiv \pi - \epsilon$ in forms that are equivalent to (Packham & Williams use B for the complex transmission coefficient and $T \equiv |B|$ for its amplitude)

$$T = \begin{cases} \tanh(\pi\eta - i\pi\tau)\tanh(\pi\eta)\, Q(i\eta,\tau) & (\ell < 1) \qquad \text{(A 1a)} \\ \tau \sinh(2\gamma\eta)\tanh(\pi\eta)\, R^2(i\eta,\tau) & (\ell > 1), \qquad \text{(A 1b)} \end{cases}$$

where

$$\tau = \tfrac{1}{2}(\pi/\gamma), \qquad \text{(A 2)}$$

$$\eta = (1/\gamma)\tanh^{-1}\begin{Bmatrix} \ell \\ 1/\ell \end{Bmatrix} \equiv \zeta/\gamma \quad (\ell \lessgtr 1), \qquad \text{(A 3)}$$

$$Q(i\eta,\tau) = e^{i\pi}\frac{M(1+\tau-i\eta,\tau)\, M(-i\eta,\tau)\, M(\tfrac{1}{2}+\tau+i\eta,\tau)\, M(\tfrac{1}{2}+i\eta,\tau)}{M(1+\tau+i\eta,\tau)\, M(i\eta,\tau)\, M(\tfrac{1}{2}+\tau-i\eta,\tau)\, M(\tfrac{1}{2}-i\eta,\tau)}, \qquad \text{(A 4a)}$$

$$R(i\eta,\tau) = e^{\frac{1}{2}i\pi}\frac{M(\tfrac{1}{2},\tau)\, M(\tau,\tau)\, M(1+\tau-i\eta)\, M(\tfrac{1}{2}+i\eta)}{M(1,\tau) M(\tfrac{1}{2}+\tau,\tau)\, M(i\eta)\, M(\tfrac{1}{2}+\tau-i\eta)}, \qquad \text{(A 4b)}$$

and M is Barnes's double gamma function, which satisfies the difference equation

$$M(z+1,\tau) = \Gamma(z/\tau)\, M(z,\tau), \qquad \text{(A 5)}$$

where Γ is the gamma function.

Packham & Williams deduce from (A 1) that

$$|T| = \begin{cases} \tanh(\pi\eta)\left\{\dfrac{\cosh(2\pi\eta) - \cos(2\pi\tau)}{\cosh(2\pi\eta) + \cos(2\pi\tau)}\right\}^{\frac{1}{2}} & (\ell < 1) \qquad \text{(A 6a)} \\ 1 & (\ell > 1) \qquad \text{(A 6b)} \end{cases}$$

† In contrast, the estimated time delay induced by San Francisco Bay at more northerly points is only three second (Miles 1972).

but do not give explicit results for the phase of T. Remarking that

$$\arg R^2(i\eta,\tau) = \arg Q(i\eta,\tau) \equiv q(\eta,\tau), \tag{A 7}$$

we obtain

$$\arg T = p(\eta,\tau)\,H(1-\ell)+q(\eta,\tau), \tag{A 8a}$$

where

$$p(\eta,\tau) = -\tan^{-1}\{\sin(2\pi\tau)/\sinh(2\pi\eta)\}, \tag{A 8b}$$

and, here and subsequently, the arctangent lies in $(-\frac{1}{2}\pi,\frac{1}{2}\pi)$. It then remains to calculate q.

Of the several forms given by Barnes (1899) for his double gamma function, we choose

$$M(z,\tau) = \frac{\exp\{a'(z/\tau)+\frac{1}{2}b'(z/\tau)^2\}}{\tau\Gamma(z)}\prod_{m=1}^{\infty}\frac{\Gamma(m\tau)\exp\{z\psi(m\tau)+\frac{1}{2}z^2\psi'(m\tau)\}}{\Gamma(z+m\tau)}, \tag{A 9}$$

where a' and b' are real parameters, the magnitudes of which are irrelevant for the present development, and ψ is the digamma function. Substituting (A 9) into (A 4a), we obtain

$$q(\eta,\tau) = \pi+2\sum_{m=0}^{\infty}\{(1-\delta_m^0)\arg\Gamma(1+m\tau+i\eta)+\arg\Gamma(m\tau+i\eta) \\ -(2-\delta_m^0)\arg\Gamma(\tfrac{1}{2}+m\tau+i\eta)\}, \tag{A 10}$$

where δ_m^0 is the Kronecker delta. Invoking the result (Abramowitz & Stegun 1965, §6.1.27)

$$\arg\Gamma(x+iy) = y\psi(x)+\sum_{n=0}^{\infty}\chi\left(\frac{y}{x+n}\right) \tag{A 11a}$$

$$= y\psi(1)-\tfrac{1}{2}\pi+\sum_{n=1}^{\infty}\chi\left(\frac{y}{n}\right)\quad (x=0), \tag{A 11b}$$

where

$$\chi(y) = y-\tan^{-1}y, \tag{A 12}$$

we transform (A 10) to

$$q(\eta,\tau) = 2\eta\left[\psi(1)-\psi(\tfrac{1}{2})+\sum_{m=1}^{\infty}\{\psi(1+m\tau)+\psi(m\tau)-2\psi(\tfrac{1}{2}+m\tau)\}\right] \\ +2\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left\{(2-\delta_m^0-\delta_n^0)\,\chi\left(\frac{\eta}{n+m\tau}\right)-(2-\delta_m^0)\,\chi\left(\frac{\eta}{\frac{1}{2}+n+m\tau}\right)\right\} \tag{A 13a}$$

$$= 4\eta\left[\ln 2+\sum_{m=1}^{\infty}\{\psi(m\tau)-\psi(\tfrac{1}{2}+m\tau)+\tfrac{1}{2}(m\tau)^{-1}\}\right] \\ +2\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(2-\delta_m^0-\delta_n^0)\,(-)^n\,\chi\left(\frac{2\eta}{n+2m\tau}\right), \tag{A 13b}$$

where (A 13b) follows from (A 13a) with the aid of standard results for the digamma function and a rearrangement of the χ summations. The convergence of the series in (A 13b) is quite adequate for a high-speed computer except in the neighbourhood of $\gamma = 0$ (see below).

The limiting results as γ tends to 0, $\pi(\epsilon \to 0)$, and 2π (half-plane) are of special interest. Letting $\gamma \downarrow 0$ (η, $\tau \uparrow \infty$) in (A 10), invoking Stirling's approximation for the gamma function, and expanding the result in inverse powers of $m\tau$ with $\eta/m\tau$ fixed, we obtain (after considerable reduction)

$$q \sim \tfrac{1}{2}\pi - \gamma\left[\tfrac{1}{4}\zeta^{-1} + \tfrac{1}{2}\zeta\sum_{m=1}^{\infty}\{(\tfrac{1}{2}\pi m)^2 + \zeta^2\}^{-1}\right] + O(\gamma^2) \tag{A 14a}$$

$$= \tfrac{1}{2}\pi - \tfrac{1}{4}(\ell + \ell^{-1})\gamma + O(\gamma^2) \quad (\gamma \downarrow 0). \tag{A 14b}$$

Expanding (A 13*b*) about $\epsilon = 0$, we obtain

$$q = \tfrac{1}{2}\epsilon\begin{Bmatrix}\ell\\ 1/\ell\end{Bmatrix} + O(\epsilon^2) \quad (\ell \lessgtr 1), \tag{A 15}$$

which, together with the corresponding expansion of (A 8*b*), yields (5.13). Setting $\tau = \frac{1}{4}$ $(\gamma = 2\pi)$ in (A 10), we obtain

$$q = \pi + 2\{\arg\Gamma(i\eta) - \arg\Gamma(1 + i\eta) + \arg\Gamma(\tfrac{1}{4} + i\eta) - \arg\Gamma(\tfrac{3}{4} + i\eta)\} \tag{A 16a}$$

$$= 2\arg\{\sec\pi(\tfrac{1}{4} - i\eta)\} \tag{A 16b}$$

$$= -\sin^{-1}\begin{Bmatrix}\ell\\ 1/\ell\end{Bmatrix} \quad (\gamma = 2\pi, \ell \lessgtr 1), \tag{A 16c}$$

where (A 16*b*) follows from (A 16*a*) with the aid of the recurrence and reflexion formulae for the gamma function. Substituting $\tau = \frac{1}{4}$ and (A 16*c*) into (A 6) and (A 8), we obtain

$$T = \begin{cases}\ell\{1 + (1 - \ell^2)^{\frac{1}{2}}\}^{-1} e^{-\frac{1}{2}i\pi} & (\gamma = 2\pi, \ell < 1) \quad \text{(A 17a)}\\ \exp\{-i\sin^{-1}(1/\ell)\} & (\gamma = 2\pi, \ell > 1). \quad \text{(A 17b)}\end{cases}$$

The only other special cases for which T may be expressed in finite terms appear to be those for which $\tau = n + \frac{1}{2}$ $(n = 1, 2, \ldots; n = 0$ is trivial). Packham & Williams (1968, § 6) reduce (A 1) to

$$T = \prod_{s=1}^{n}\frac{\sin 2(s\gamma - i\zeta)}{\sin 2(s\gamma + i\zeta)} \quad (\pi/\gamma = 2n + 1) \tag{A 18}$$

and deduce that $|T| = 1$ for these corners. The corresponding phase shift is given by

$$\arg T = -2\sum_{s=1}^{n}\tan^{-1}\left\{\left(\frac{2\ell}{1 + \ell^2}\right)\cot\left(\frac{2s\pi}{2n + 1}\right)\right\} \quad (n = 1, 2, \ldots). \tag{A 19}$$

The result (A 8) is plotted in figure 3. The quantity p is oscillatory in $\gamma > \frac{1}{2}\pi$ and has an oscillatory singularity as $\epsilon \uparrow \pi$, for which reason $\arg T$ is not plotted in $\epsilon > \frac{1}{2}\pi$ for $\ell < 1$. Numerical values in this domain may be obtained by calculating p from (A 8*b*) and involving the fact that $q \equiv \arg T$ for $\ell > 1$ is invariant under the transformation $\ell \leftrightarrow 1/\ell$.

REFERENCES

ABRAMOWITZ, M. & STEGUN, I. 1965 *Handbook of Mathematical Functions*. Washington: National Bureau of Standards.

BARNES, E. W. 1899 The genesis of the double gamma function. *Proc. Lond. Math. Soc.* **31** (1), 358–381.

BUCHWALD, V. T. 1971 The diffraction of tides by a narrow channel. *J. Fluid Mech.* **46**, 501–511.

LAMB, H. 1932 *Hydrodynamics*. Cambridge University Press.

LIGHTHILL, M. J. 1957 The fundamental solution for small steady three-dimensional disturbances in a two-dimensional parallel shear flow. *J. Fluid Mech.* **3**, 113–144.

MILES, J. W. 1972 A note on tidal delay by inland seas and harbors. *Geophys. Fluid Dyn.* **3**, 27–34.

MOORE, D. W. 1968 Planetary-gravity waves in an equatorial ocean. Ph.D. Thesis, Harvard University.

MUNK, W., SNODGRASS, F. & WIMBUSH, M. 1970 Tides off-shore: transition from California coastal to deep-sea waters. *Geophys. Fluid Dyn.* **1**, 161–235.

PACKHAM, B. & WILLIAMS, W. 1968 Diffraction of Kelvin waves at a sharp bend. *J. Fluid Mech.* **34**, 517–529.

PINSENT, H. G. 1972 Kelvin wave attenuation along nearly straight boundaries. *J. Fluid Mech.* **53**, 273–286.

SMITH, R. 1972 Nonlinear Kelvin and continental-shelf waves. *J. Fluid Mech.* **52**, 379–392.

Note added in proof. Pinsent (1972) has recently carried out a calculation to second order of the effects on a Kelvin wave of both changing depth along the coast and departure of the coast from a straight line; however, his result for the 'transmitted Kelvin wave', equation (3.11), is valid only for changes of compact support in the sense that the changes in depth and coastline displacement must have been completely traversed at the point of observation. Pinsent also (in his §5.2) calculates local amplitude and phase changes and compares his results for the California coastline with the data of Munk *et al.* (1970). He obtains similar trends in the coastal variation of both tidal amplitude and tidal phase; however, he appears to have fitted the coastline at the inner, rather than the outer, edge of the continental shelf and to have neglected $\sin\theta$ in calculating the Coriolis parameter.